

ENGEL-TYPE SUBGROUPS AND LENGTH PARAMETERS OF FINITE GROUPS

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ABSTRACT. Let g be an element of a finite group G . For a positive integer n , let $E_n(g)$ be the subgroup generated by all commutators $[...[[x, g], g], \dots, g]$ over $x \in G$, where g is repeated n times. By Baer's theorem, if $E_n(g) = 1$, then g belongs to the Fitting subgroup $F(G)$. We generalize this theorem in terms of certain length parameters of $E_n(g)$. For soluble G we prove that if, for some n , the Fitting height of $E_n(g)$ is equal to k , then g belongs to the $(k+1)$ th Fitting subgroup $F_{k+1}(G)$. For nonsoluble G the results are in terms of nonsoluble length and generalized Fitting height. The generalized Fitting height $h^*(H)$ of a finite group H is the least number h such that $F_h^*(H) = H$, where $F_0^*(H) = 1$, and $F_{i+1}^*(H)$ is the inverse image of the generalized Fitting subgroup $F^*(H/F_i^*(H))$. Let m be the number of prime factors of $|g|$ counting multiplicities. It is proved that if, for some n , the generalized Fitting height of $E_n(g)$ is equal to k , then g belongs to $F_{f(k,m)}^*(G)$, where $f(k, m)$ depends only on k and m . The nonsoluble length $\lambda(H)$ of a finite group H is defined as the minimum number of nonsoluble factors in a normal series each of whose factors either is soluble or is a direct product of nonabelian simple groups. It is proved that if $\lambda(E_n(g)) = k$, then g belongs to a normal subgroup whose nonsoluble length is bounded in terms of k and m . We also state conjectures of stronger results independent of m and show that these conjectures reduce to a certain question about automorphisms of direct products of finite simple groups.

1. INTRODUCTION

By the well-known theorem of Baer [2, Satz III.6.15], an element g of a finite group G belongs to the Fitting subgroup $F(G)$ if and only if it is a left-Engel element, that is, if $[x, g, g, \dots, g] = 1$ for all $x \in G$, where g is repeated in the commutator sufficiently many times. (Throughout the paper, we use the left-normed simple commutator notation $[a_1, a_2, a_3, \dots, a_r] = [...[[a_1, a_2], a_3], \dots, a_r]$.) In this paper we generalize this result using the subgroups

$$E_n(g) = \langle [x, \underbrace{g, \dots, g}_n] \mid x \in G \rangle.$$

Recall that the Fitting series is defined starting from $F_0(G) = 1$, and then by induction $F_{k+1}(G)$ is the inverse image of $F(G/F_k(G))$. Our first result is about soluble groups.

Theorem 1.1. *Let g be an element of a finite soluble group G , and n a positive integer. If the Fitting height of $E_n(g)$ is equal to k , then g belongs to $F_{k+1}(G)$.*

For nonsoluble finite groups we prove similar results in terms of the nonsoluble length and generalized Fitting height of $E_n(g)$. We recall the relevant definitions. The generalized

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Fitting subgroup $F^*(G)$ is the product of the Fitting subgroup $F(G)$ and all subnormal quasisimple subgroups (here a group is quasisimple if it is equal to its derived subgroup and its quotient by the centre is a non-abelian simple group). Then the *generalized Fitting series* of G is defined starting from $F_0^*(G) = 1$, and then by induction $F_{i+1}^*(G)$ is the inverse image of $F^*(G/F_i^*(G))$. The least number h such that $F_h^*(G) = G$ is naturally defined to be the *generalized Fitting height* $h^*(G)$ of G . Clearly, if G is soluble, then $h^*(G) = h(G)$ is the ordinary Fitting height of G .

Theorem 1.2. *Let m and n be positive integers, and let g be an element of a finite group G whose order $|g|$ is equal to the product of m primes counting multiplicities. If the generalized Fitting height of $E_n(g)$ is equal to k , then g belongs to $F_j^*(G)$ for some $j \leq ((k+1)m(m+1)+2)(k+3)/2$.*

This theorem follows from Theorem 1.1 on soluble groups and Theorem 1.3 below concerning another length parameter. Namely, the *nonsoluble length* $\lambda(G)$ of a finite group G is defined as the minimum number of nonsoluble factors in a normal series each of whose factors either is soluble or is a direct product of nonabelian simple groups. (In particular, the group is soluble if and only if its nonsoluble length is 0.) Bounds for the nonsoluble length or/and generalized Fitting height of a finite group G greatly facilitate using the classification of finite simple groups (and are themselves often obtained by using the classification). Most notably such bounds were used in the reduction of the Restricted Burnside Problem to soluble and nilpotent groups in the Hall–Higman paper [1]. Such bounds also find applications in the study of profinite groups. Examples include Wilson’s reduction of the problem of local finiteness of periodic profinite groups to pro- p groups in [4] and our recent paper [3] on similar problems. (Both the Restricted Burnside Problem and the problem of local finiteness of periodic profinite groups were solved by Zelmanov [5, 6, 7].)

Similarly to the generalized Fitting series, we can define terms of the ‘upper nonsoluble series’: let $R_i(G)$ be the maximal normal subgroup of G that has nonsoluble length i (so that, in particular, $R_0(G)$ is the soluble radical of G).

Theorem 1.3. *Let m and n be positive integers, and let g be an element of a finite group G whose order $|g|$ is equal to the product of m primes counting multiplicities. If the nonsoluble length of $E_n(g)$ is equal to k , then g belongs to $R_j(G)$ for some $j \leq (k+1)m(m+1)/2$.*

Note that $E_n(g)$ is *not* a subgroup of the type $[\dots[[G, g], g], \dots, g]$ formed by taking successive commutator subgroups, which is subnormal. But $E_n(g)$ is not subnormal in general.

Our results on nonsoluble groups depend on the classification of finite simple groups in so far as the validity of the Schreier conjecture on solubility of the group of outer automorphisms of a finite simple group. In Theorems 1.2 and 1.3, nonsoluble length and generalized Fitting height have bounds depending both on the parameters of $E_n(g)$ and on the number of prime factors in the order of g . We conjecture that stronger results may hold, not depending on the order of g . Namely, we conjecture that if, for an element g of a finite group G the nonsoluble length of $E_n(g)$ is equal to k , then g belongs to $R_k(G)$. We also conjecture that if the generalized Fitting height of $E_n(g)$ is equal to k , then g belongs to $F_{k+1}^*(G)$. These conjectures can be reduced to certain questions about automorphisms of direct products of nonabelian simple groups. The questions, though, seem rather difficult and remain open. We discuss them in §7.

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2. PRELIMINARIES AND PROOF OF THEOREM 1.1

We notice nice ‘radical’ properties of the subgroups $F_i(G)$, $F_i^*(G)$, and $R_i(G)$, which will be often used without special references. Namely, it follows from the definitions that if N is a normal subgroup of G , then

$$F_i(N) = N \cap F_i(G), \quad F_i^*(N) = N \cap F_i^*(G), \quad R_i(N) = N \cap R_i(G).$$

For the Fitting subgroups we also have the inclusions $F_i(G) \cap H \leq F_i(H)$ for any, not necessarily normal, subgroup $H \leq G$. However, similar inclusions do not hold in general for the subgroups $F_i^*(G)$ and $R_i(G)$.

As a consequence we have the following.

Lemma 2.1. *Let N be a subnormal subgroup of a finite group G . Then*

(a) *the Fitting height (when G is soluble), the generalized Fitting height, and the non-soluble length of N do not exceed the corresponding parameters of G , and*

(b) *the Fitting height (when N is soluble), the generalized Fitting height, and the non-soluble length of the normal closure $\langle N^G \rangle$ are equal to the corresponding parameters of N .*

It also follows from the definitions that

$$F_i(G/N) \geq F_i(G)N/N, \quad F_i^*(G/N) \geq F_i^*(G)N/N, \quad R_i(G/N) \geq R_i(G)N/N$$

for any normal subgroup N of G .

When we consider a group A acting by automorphisms on a group G , we regard A as a subgroup of the natural semidirect product GA , so that we can form the mutual commutator subgroup $[G, A]$ and use the centralizer notation for the fixed-point subgroup $C_G(A)$. Throughout the paper we use without special references the well-known properties of coprime actions: if α is an automorphism of a finite group G of coprime order, $(|\alpha|, |G|) = 1$, then $C_{G/N}(\alpha) = C_G(\alpha)N/N$ for any α -invariant normal subgroup N , the equality $[G, \alpha] = [[G, \alpha], \alpha]$ holds, and if G is in addition abelian, then $G = [G, \alpha] \times C_G(\alpha)$.

We generalize slightly the notation introduced above. If H is a g -invariant subgroup of a group G , where $g \in G$ or $g \in \text{Aut } G$, then let

$$E_{H,n}(g) = \langle [h, \underbrace{g, g, \dots, g}_n] \mid h \in H \rangle.$$

Thus, $E_n(g) = E_{G,n}(g)$ when $g \in G$ and it is clear from the context which group G is involved. It is clear that $C_G(g)$ normalizes $E_{G,n}(g)$.

We are now going to prove Theorem 1.1. The proof reduces to the following proposition.

Proposition 2.2. *Let α be an automorphism of a finite soluble group G such that $G = [G, \alpha]$. Let n be a positive integer. Then $E_{G,n}(\alpha) = G$.*

Proof. Let $E = E_{G,n}(\alpha)$ for brevity. Let G be a counterexample of minimal order, and let M be a minimal α -invariant normal subgroup of G . Since the image of E is obviously equal to the similar subgroup constructed for G/M , we have $G = ME$ and $M \not\leq E$. Then $M \cap E = 1$ by minimality of M because M is abelian and $G = ME$.

Suppose that $C_M(\alpha) \neq 1$. Since $C_M(\alpha)$ normalizes E , we have $[C_M(\alpha), E] \leq M \cap E = 1$. Then $C_M(\alpha)$ is central in G since M is abelian and $G = EM$. By minimality, $C_M(\alpha) = M$ and $G = M \times E$. This contradicts the hypothesis $G = [G, \alpha]$.

Thus, $C_M(\alpha) = 1$. Then the mapping $m \rightarrow [m, \alpha]$ of M into M is injective, and therefore surjective, since M is finite. Hence every element of M has the form $[m, \alpha]$, and therefore also the form $[m, \alpha, \alpha, \dots, \alpha]$ with α repeated n times. Then $M \leq E$, which contradicts the equation $E \cap M = 1$. \square

Proof of Theorem 1.1. Recall that g is an element of a finite soluble group G , and the Fitting height of $E_n(g)$ is equal to k ; we need to show that g belongs to $F_{k+1}(G)$. Consider the subgroups

$$[\dots[[G, g], g], \dots, g]$$

formed by taking successive commutator subgroups; in particular, these subgroups are subnormal in G . Let H be the smallest of the subgroups $[\dots[[G, g], g], \dots, g]$. Note that if $H = 1$, then g is a left-Engel element and therefore $g \in F(G)$ and there is nothing to prove. In any case, $H = [H, g]$.

Let $N = \langle H^G \rangle$ be the normal closure of H . By construction, the image of g in G/N is a left-Engel element and therefore belongs to the Fitting subgroup of G/N . If we prove that the Fitting height of N is at most k , then N , as a normal subgroup, is contained in $F_k(G)$ and then $g \in F_{k+1}(G)$, as required. Since H is subnormal in G , the Fitting height of its normal closure N is the same as that of H by Lemma 2.1(b). Thus, it suffices to show that the Fitting height of H is at most k .

Consider the automorphism α induced on H by g acting by conjugation. The subgroup $E_{H,n}(\alpha)$ is clearly contained in $E_n(g)$ and therefore has Fitting height at most k . Since $[H, \alpha] = H$, by Proposition 2.2 we obtain $H = E_{H,n}(\alpha)$; hence the result. \square

3. DIRECT PRODUCTS OF NONABELIAN FINITE SIMPLE GROUPS

We begin with elementary consequences of the Schreier conjecture, which may well be known.

Lemma 3.1. *If a finite group G has a normal series all of whose factors are nonabelian simple groups, then G is a direct product of nonabelian simple groups.*

Proof. Let

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_{j-1} \triangleleft G_j = G$$

be a normal series all of whose factors G_i/G_{i-1} are nonabelian simple groups. We use induction on the length of the series j , the basis of which is obvious. For $j > 1$, the quotient G/G_1 is a direct product of nonabelian simple groups by the induction hypothesis. The quotient $G/C_G(G_1)$ embeds in the automorphism group of G_1 . Since the group of outer automorphisms of G_1 is soluble by the Schreier conjecture and G has no soluble homomorphic images by the Jordan–Hölder theorem, we must have $G = G_1 C_G(G_1) = G_1 \times C_G(G_1)$, since $G_1 \cap C_G(G_1) = 1$. The result follows. \square

We shall use without special references the well-known fact that in any direct product $S_1 \times \dots \times S_r$ of nonabelian finite simple groups the only normal subgroups are products $S_{i_1} \times \dots \times S_{i_t}$ of some of the factors.

Recall that a subgroup H of a direct product $G_1 \times \dots \times G_r$ is called a subdirect product of the groups G_1, \dots, G_r if the natural projection of H onto every G_i is equal to G_i .

Lemma 3.2. *Let $G = S_1 \times \dots \times S_r$ be a direct product of isomorphic nonabelian finite simple groups. Let $H \leq G$ be a subdirect product of the same groups. Then $H = H_1 \times \dots \times H_u$, where every H_i is isomorphic to S_1 .*

Proof. Let T_i denote the kernel of the i th projection of H . Then intersections of some of the T_i form a series as in Lemma 3.1. \square

For the rest of the section we shall work under the following setting.

Hypothesis 3.3. Let $S = S_1 \times \cdots \times S_r$ be a direct product of r isomorphic finite non-abelian simple groups and let φ be the natural automorphism of S of order r that regularly permutes the S_i .

Note that S has no proper normal φ -invariant subgroups. In particular, $[S, \varphi] = S$.

Let $D = C_S(\varphi)$ be the diagonal. More generally, a subgroup of S that is the diagonal (with respect to φ) in the product of some of the S_i (not necessarily of all) is called a d -subgroup. More precisely, a d -subgroup has the form

$$\{(\dots, a_{i_1}, \dots, a_{i_2}, \dots, a_{i_l}, \dots) \mid a_{i_s} = a_{i_1}^{\varphi^{i_s - i_1}}\},$$

where dots denote segments of 1s (possibly empty). In cases where we want to indicate the set of indices $I = \{i_1, \dots, i_l\}$ of nontrivial coordinates in a d -subgroup, we call it a $d(I)$ -subgroup.

Lemma 3.4. Under Hypothesis 3.3, suppose that K is the $d(I)$ -subgroup for some I . Then $C_S(K)$ is the product of S_i with $i \notin I$ and $N_S(K) = K \times C_S(K)$.

Proof. This follows from the fact that $Z(S_i) = 1$. The part about $C_S(K)$ is clear. If an element b of S has different coordinates $b_u \neq b_v$ for $u, v \in I$, then there is $x \in K$ with nontrivial coordinates a such that $a^{b_u} \neq a^{b_v}$, so that $b \notin N_S(K)$. \square

Lemma 3.5. Under Hypothesis 3.3, suppose that K is a d -subgroup and $x \in S$ is such that K^x is also a d -subgroup. Then actually $x \in N_S(K)$.

Proof. This follows from the fact that conjugation by $x \in S$ does not change the set of indices of nontrivial coordinates. \square

Lemma 3.6. Under Hypothesis 3.3, let a subgroup H of S be isomorphic to S_1 and suppose that H is normalized by the diagonal D . Then H is a d -subgroup.

Proof. It is sufficient to show that if H has nontrivial projections on some two subgroups S_i and S_j , then the projection P of H onto $S_i \times S_j$ is equal to the $d(\{i, j\})$ -subgroup K . If this is not the case, then $P \cap K = 1$, since P is normalized by K by hypothesis and $K \cong P \cong S_1$. Then $PK = S_i \times S_j$ and therefore P is normal in $S_i \times S_j$. Being a proper subgroup, it must be one of the factors, which contradicts the choice of S_i, S_j . \square

Lemma 3.7. Assume Hypothesis 3.3 with $r = |\varphi|$ being a prime. If H is a φ -invariant subgroup of S containing the diagonal D , then either $H = D$ or $H = S$.

Proof. Since $H \geq D$, by Lemma 3.2 we have $H = H_1 \times \cdots \times H_l$, where every H_i is isomorphic to S_1 . Since $D \leq H$ normalizes every H_i , by Lemma 3.6 H_i is a d -subgroup. We assume that $H \neq D$ and prove that $H = S$. Choose $H_i \neq D$. We claim that in fact $H_i = S_j$ for some i, j , which will imply that $H = S$ due to the φ -invariance. Suppose that H_i has more than one nontrivial projections onto the S_i — but necessarily less than $|\varphi|$, since $H_i \neq D$ and H_i is a d -subgroup. Since $r = |\varphi|$ is a prime, then we can choose a power of φ such that $H_i^{\varphi^w} \neq H_i$ and both $H_i^{\varphi^w}$ and H_i have nontrivial projections onto the same S_k . The former implies that $H_i^{\varphi^w} = H_j \neq H_i$, so that $[H_i, H_i^{\varphi^w}] = 1$, while the latter implies that $[H_i, H_i^{\varphi^w}] \neq 1$, a contradiction. \square

Lemma 3.8. Assume Hypothesis 3.3 with $r = |\varphi|$ being a prime. Let n be a positive integer and let $F = \langle [x, \underbrace{\varphi, \dots, \varphi}_n] \mid x \in S_i \rangle$. Then $F = S$.

Proof. First note that $[F, \varphi] \neq 1$. Indeed, choose some element $x \in S_1$ of prime order $q \neq r$. Since $Q = \langle x^{\langle \varphi \rangle} \rangle$ is an abelian φ -invariant q -group, we have

$$1 \neq [Q, \varphi] = [[Q, \varphi], \varphi] = \{[x, \underbrace{\varphi, \varphi, \dots, \varphi}_k] \mid k \geq n+1\}.$$

In particular, $1 \neq [x, \underbrace{\varphi, \varphi, \dots, \varphi}_{n+1}] \in [F, \varphi]$. It follows from the definition that F is a φ -invariant subgroup, which is also normalized by D . Then FD is a φ -invariant subgroup properly containing D and therefore $FD = S$ by Lemma 3.7. As a result, F is normal in S . Being nontrivial and φ -invariant, it must coincide with S . \square

Lemma 3.9. *Assume Hypothesis 3.3. Let n be a positive integer and let $E = E_{S,n}(\varphi) = \langle [x, \underbrace{\varphi, \dots, \varphi}_n] \mid x \in S \rangle$. Then $E = S$.*

Proof. Let φ^k be of prime order. Then $G = \langle [g, \underbrace{\varphi^k, \dots, \varphi^k}_n] \mid g \in S_i, i = 1, \dots, r \rangle$ by Lemma 3.8 applied to each orbit of φ^k . It remains to observe that $[g, \underbrace{\varphi^k, \dots, \varphi^k}_n] \in E$ for $g \in S_i$. Indeed, $A = \langle g^{\langle \varphi \rangle} \rangle$ is an abelian φ -invariant group, in which repeated application of the formulae $[b, \varphi^{i+1}] = [b, \varphi^i][b, \varphi]$ and $[bc, \varphi^i] = [b, \varphi^i][c, \varphi^i]$ for $b, c \in A$ expresses the commutator $[g, \underbrace{\varphi^k, \dots, \varphi^k}_n]$ in terms of commutators $[x, \underbrace{\varphi, \dots, \varphi}_n]$ for $x \in A$. \square

4. ORBITS IN SOME TRANSITIVE PERMUTATIONAL ACTIONS

We shall need several lemmas on transitive permutational actions of certain finite groups G concerning the existence of exact (regular) orbits of an element $g \in G$. Rather than speaking about orbits, we prefer to state these lemmas in terms of intersections of $\langle g \rangle$ with a conjugate of the stabilizer of a point $H \leq G$. The following lemma is quite elementary.

Lemma 4.1. *Let g be an element, and H a subgroup of a group G . Suppose that g is contained in a subgroup $G_0 \leq G$ and $\langle g \rangle$ has trivial intersection with $(H \cap G_0)^x$ for some $x \in G_0$. Then $H^x \cap \langle g \rangle = 1$.*

Proof. Let $H_0 = H \cap G_0$. Then

$$H^x \cap \langle g \rangle = H^x \cap G_0 \cap \langle g \rangle = (H \cap G_0)^x \cap \langle g \rangle = H_0^x \cap \langle g \rangle = 1. \quad \square$$

Lemma 4.2. *Let $S = S_1 \times \dots \times S_l$ be a direct product of finitely many nonabelian finite simple groups. Let φ be an automorphism of S such that every φ -orbit of the permutational action on $\{S_1, \dots, S_l\}$ has $|\varphi|$ elements. Let H be a subgroup of $S\langle \varphi \rangle$ such that $S \not\leq H$. Then there is $x \in S$ such that $H^x \cap \langle \varphi \rangle = 1$.*

Proof. We proceed by induction on $|S\langle \varphi \rangle|$. As a basis of induction we can consider the case where $|\varphi|$ is a prime, where the result is obvious. Indeed, if $\varphi \in \bigcap_{x \in S} H^x$, then, since $\bigcap_{x \in S} H^x$ is S -invariant,

$$S = [S, \varphi] \leq [S, \bigcap_{x \in S} H^x] \leq H,$$

a contradiction with hypothesis. Next, if the order of φ is divisible by p^2 for some prime p , then we can work with φ^p , which also satisfies the condition that all its orbits are of length $|\varphi^p|$. By induction applied to $S\langle \varphi^p \rangle$ and $H_0 = H \cap S\langle \varphi^p \rangle$ we obtain $x \in S$ such that $H_0^x \cap \langle \varphi^p \rangle = 1$, whence $H^x \cap \langle \varphi^p \rangle = 1$ by Lemma 4.1 and then $H^x \cap \langle \varphi \rangle = 1$. So

we can assume that the order of φ is square-free and is divisible by at least two primes. Let p be a prime divisor of $|g|$. Let $\beta = \varphi^p$ and let α be an element of order p in $\langle \varphi \rangle$. By induction there exists $x \in S$ such that $(H \cap S\langle \beta \rangle)^x \cap \langle \beta \rangle = 1$, whence $H^x \cap \langle \beta \rangle = 1$. If $\alpha \notin H^x$, then the proof is complete. Therefore we can assume that $\alpha \in H^x$. Replacing H by H^x we now assume that $H \cap \langle \beta \rangle = 1$ and $\alpha \in H$.

We can choose a φ -orbit such that the product T of the simple factors in this orbit satisfies $T \not\leq H$. If $T \neq S$, then by induction there is $x \in T$ such that $(H \cap T\langle \varphi \rangle)^x \cap \langle \varphi \rangle = 1$; then $H^x \cap \langle \varphi \rangle = 1$ by Lemma 4.1. Thus, we can assume that S is a product over one φ -orbit. Let $C = C_S(\beta)$. Since $H^x \cap \langle \beta \rangle = 1$ for all $x \in C$, we can assume that $\alpha^x \in H$ for all $x \in C$. Since $\alpha^x = \alpha[\alpha, x]$ and $\alpha \in H$, we deduce that $[C, \alpha] \leq H$. Since C is the direct product of d -subgroups corresponding to β -orbits, and these d -subgroups are regularly permuted by α , it follows that $C = [C, \alpha]$. In particular, the diagonal D of S is contained in H and so $H \cap S$ is a subdirect product of the S_i . Hence $H \cap S$ is a direct product of subgroups H_i isomorphic to S_1 by Lemma 3.2.

If $SH \neq S\langle \varphi \rangle$, then $SH = S\langle \varphi^t \rangle$ for some $t > 1$. In this case we can apply the induction hypothesis to $S\langle \varphi^t \rangle$, find $x \in S$ such that $H^x \cap \langle \varphi^t \rangle = 1$, and then also $H^x \cap \langle \varphi \rangle = H^x \cap S\langle \varphi^t \rangle \cap \langle \varphi \rangle = H^x \cap \langle \varphi^t \rangle = 1$, since $S \cap \langle \varphi \rangle = 1$. Thus, we assume that $SH = S\langle \varphi \rangle$. Then $s\varphi = h$ for some $s \in S$ and $h \in H$. Note that h induces the same permutation on $\{S_1, \dots, S_l\}$ as φ .

Since $D \leq H$ normalizes every H_i , by Lemma 3.6 we obtain that H_i is a d -subgroup. We claim that in fact $H_i = S_j$ for some j . Indeed, choose an element $a_1 \in S_1$ such that $a_1 \neq a_1^{-1}$. By assumption, $H^{a_1^{-1}} \cap \langle \varphi \rangle \neq 1$, so that there is a nontrivial element $\psi \in \langle \varphi \rangle$ such that $\psi^{a_1} \in H$. Then for every i we have $H_i^{\psi^{a_1}} = H_j$ for some $j = j(i)$. Since H_i^ψ is also a d -subgroup and $(H_i^\psi)^{[\psi, a_1]} = H_i^{\psi^{a_1}} = H_j$, by Lemma 3.5 we obtain that $[\psi, a_1]$ normalizes $H_i^\psi = H_j$. Thus, ψ permutes the subgroups H_u . It follows that $H_j^{[\psi, a_1]} = H_j$ for all j . Since $D \leq H$, there is a d -subgroup H_j with nontrivial projections onto at least one of the factors S_1 or $S_k = S_1^\psi \neq S_1$, which contain the nontrivial projections a_1 and $(a_1^{-1})^\psi$ of $[\psi, a_1]$. Since $a_1 \neq a_1^{-1}$, it follows from Lemma 3.4 that $H_j = S_1$ or $H_j = S_k$.

The latter implies that $S \leq H$ due to the invariance under $h \in H$, which induces the same permutation on the S_u as φ . This contradicts the hypothesis. \square

Lemma 4.3. *Let $S = S_1 \times \dots \times S_r$ be a direct product of r isomorphic nonabelian finite simple groups, and let φ be an automorphism of S that transitively permutes the S_i . Suppose that for some prime p the stabilizer of S_1 in $\langle \varphi \rangle$ is a p -subgroup, possibly trivial. Let H be a subgroup of $S\langle \varphi \rangle$ such that $S \not\leq H$. Then there is $x \in S$ such that $H^x \cap \langle \varphi \rangle = 1$.*

Note that we do not exclude the case of $r = 1$, when of course φ is an automorphism of $S = S_1$ of prime-power order.

Remark 4.4. The condition in Lemma 4.3 that the stabilizer of S_1 in $\langle \varphi \rangle$ is a p -subgroup is essential. The smallest example is given by the alternating group $S_1 = \text{Alt}_5$, its automorphism φ induced by conjugation by an element of order 6 in the ambient symmetric group Sym_5 , and H being a stabilizer of a point in the natural representation of Sym_5 on 5 points.

Proof of Lemma 4.3. We proceed by induction on $|S\langle \varphi \rangle|$. As a basis of induction we can consider the case where $|\varphi|$ is a prime, where the result is obvious. Indeed, if $\varphi \in \bigcap_{x \in S} H^x$, then, since $\bigcap_{x \in S} H^x$ is S -invariant, $S = [S, \varphi] \leq [S, \bigcap_{x \in S} H^x] \leq H$, a contradiction with hypothesis.

Next, if the order of φ is divisible by q^2 for some prime q , then we can work with φ^q in place of φ . There is an orbit (possibly one-element) of the permutation induced by φ^q on the set $\{S_1, \dots, S_r\}$ such that the product T over this orbit satisfies $T \not\leq H$. Then the group $T\langle\varphi^q\rangle$ satisfies the hypotheses of the lemma with $H_0 = H \cap T\langle\varphi^q\rangle$ in place of H . By induction we find $x \in T$ such that $H_0^x \cap \langle\varphi^q\rangle = 1$, whence $H_0^x \cap \langle\varphi\rangle = 1$, and then $H^x \cap \langle\varphi\rangle = 1$ by Lemma 4.1. So we can assume that the order of φ is square-free and is divisible by at least two primes.

If the Sylow p -subgroup of $\langle\varphi\rangle$ is trivial, then we get the result by Lemma 4.2. So let α be an element of order p which generates the stabilizer of S_1 in $\langle\varphi\rangle$, and let $\beta = \varphi^p$. By induction there exists $x \in S$ such that $(H \cap S\langle\beta\rangle)^x \cap \langle\beta\rangle = 1$, whence $H^x \cap \langle\beta\rangle = 1$ by Lemma 4.1. If $\alpha \notin H^x$, then the proof is complete; so we can assume that $\alpha \in H^x$. Replacing H by H^x we now assume that $H \cap \langle\beta\rangle = 1$ and $\alpha \in H$.

Let $D = C_S(\beta)$. Since $H^x \cap \langle\beta\rangle = 1$ for all $x \in D$, we can assume that $\alpha^x \in H$ for all $x \in D$. Since $\alpha^x = \alpha[\alpha, x]$ and $\alpha \in H$, we deduce that $D = [D, \alpha] \leq H$. Hence $H \cap S$ is a subdirect product of the S_i . By Lemma 3.2 we obtain that $H \cap S$ is a direct product of subgroups H_i isomorphic to S_1 .

If $SH \neq S\langle\varphi\rangle$, then $SH = S\langle\varphi^t\rangle$ for some $t > 1$. We can choose an orbit of the permutation induced by φ^t on the set $\{S_1, \dots, S_r\}$ such that the product T of the simple factors in this orbit satisfies $T \not\leq H$. The group $T\langle\varphi^t\rangle$ satisfies the hypotheses of the lemma with $H_0 = H \cap T\langle\varphi^t\rangle$ in place of H . By induction we find $x \in T$ such that $H_0^x \cap \langle\varphi^t\rangle = 1$, whence $H^x \cap \langle\varphi^t\rangle = 1$ by Lemma 4.1, and then $H^x \cap \langle\varphi\rangle = H^x \cap S\langle\varphi^t\rangle \cap \langle\varphi\rangle = H^x \cap \langle\varphi^t\rangle = 1$, since $S \cap \langle\varphi\rangle = 1$. Thus, we assume that $SH = S\langle\varphi\rangle$. Then $s\varphi = h$ for some $s \in S$ and $h \in H$. Note that h induces the same permutation on $\{S_1, \dots, S_r\}$ as φ .

Since $D \leq H$, it follows that D normalizes every H_i , and by Lemma 3.6 we obtain that every H_i is a d -subgroup. If $D = H \cap S$, then D is normalized by $s\varphi = h \in H$. Since D is also normalized by φ , we obtain that D is normalized by s , whence $s \in D$ by Lemma 3.4. Since $D \leq H$, we obtain that $\varphi \in H$, a contradiction with our assumption. Therefore we can assume that there is $H_i \neq D$.

We claim that in fact $H_i = S_j$ for some j . Indeed, choose an element $a_1 \in S_1$ such that $a_1 \neq a_1^{-1}$. By assumption, $H^{a_1^{-1}} \cap \langle\varphi\rangle \neq 1$, so that there is a nontrivial element $\psi \in \langle\varphi\rangle$ such that $\psi^{a_1} \in H$. If $\psi \notin \langle\alpha\rangle$, then we argue in the same fashion as in the proof of Lemma 4.2. Namely, H_i^ψ is a d -subgroup for every i and $(H_i^\psi)^{[\psi, a_1]} = H_i^{\psi^{a_1}} = H_j$; hence $(H_i^\psi)^{[\psi, a_1]} = H_i^\psi$ by Lemma 3.5; therefore $H_j^{[\psi, a_1]} = H_j$ for all j . Since $D \leq H$, there is a d -subgroup H_j with nontrivial projections onto at least one of the factors S_1 or $S_k = S_1^\psi \neq S_1$, which contain a_1 and $(a_1^{-1})^\psi$. Since $a_1 \neq a_1^{-1}$, it follows from Lemma 3.4 that $H_j = S_1$ or $H_j = S_k$, which leads to $S \leq H$, a contradiction.

It remains to consider the case where $1 \neq H^{a_1^{-1}} \cap \langle\varphi\rangle \leq \langle\alpha\rangle$ for all $a_1 \in S_1$ such that $|a_1| \neq 2$. Since such elements a_1 generate S_1 and α is a nontrivial automorphism of S_1 , there is $a_1 \in S_1$ such that $a_1 \neq a_1^\alpha$. Since $\alpha \in H$ and $\alpha^{a_1} \in H$ by our assumption, we obtain a nontrivial element $[\alpha, a_1] \in S_1 \cap H$. Since $H \cap S$ is a direct product of d -subgroups H_i , we must have $S_1 = H_j$ for some j .

In any case, we have $H_i = S_j$ for some i, j . This implies that $H = S$ due to the invariance under $h \in H$, which induces the same permutation on the S_u as φ . This contradicts the hypothesis. \square

Let N be a normal subgroup of a group G , and A a subgroup of G/N . We shall be saying for brevity that B covers A if $BN/N \geq A$.

The following proposition is the main result of this section.

Proposition 4.5. *Let $G\langle g \rangle$ be a finite group with a normal subgroup G and a cyclic subgroup $\langle g \rangle$, and let R be the soluble radical of G . Let $\langle g_0 \rangle = C_{\langle g \rangle}(G/R)$, so that the image \bar{g} of g in $\langle g \rangle / \langle g_0 \rangle$ is the automorphism of G/R induced by conjugation by g . Suppose that the following conditions hold:*

- (1) G is a minimal g -invariant subgroup covering G/R ;
- (2) $G \cap \langle g \rangle \leq C_{\langle g \rangle}(G/R)$;
- (3) $G/R = S_1 \times \cdots \times S_r$ is a direct product of isomorphic nonabelian finite simple groups, which are transitively permuted by \bar{g} ;
- (4) for some prime p the stabilizer of S_1 in $\langle \bar{g} \rangle$ is a p -subgroup, possibly trivial.

Then for any subgroup $H \leq G\langle g \rangle$ such that $G \not\leq H$ there is $z \in G$ such that $H^z \cap \langle g \rangle \leq \langle g_0 \rangle$.

For brevity we shall refer to orbits of elements of $G\langle g \rangle$ on $\{S_i\}$ meaning orbits in the set $\{S_1, \dots, S_r\}$ of the corresponding induced permutations.

Proof. Induction on $|G\langle g \rangle|$. As a basis of induction we can take the case $R = 1$. In this case, $\langle g_0 \rangle$ is a central subgroup, and in the quotient $G\langle g \rangle / \langle g_0 \rangle$ the image of H does not contain the image of G . Indeed, otherwise $G \leq H\langle g_0 \rangle$ and then $G = [G, G] \leq [H\langle g_0 \rangle, H\langle g_0 \rangle] = [H, H]$, contrary to the assumption $G \not\leq H$. Thus, due to conditions (3), (4), we can apply Lemma 4.3 to the group $G\langle g \rangle / \langle g_0 \rangle$ and find $z \in G$ such that $H^z \cap \langle g \rangle \leq \langle g_0 \rangle$, as required.

Now let $R \neq 1$ and let M be a minimal g -invariant abelian normal subgroup of G . If $G \not\leq MH$, then we can apply induction to $G\langle g \rangle / M$. Indeed, since $M \leq R$, conditions (1), (3), (4) obviously hold for the images in $G\langle g \rangle / M$, and the image of $\langle g_0 \rangle$ is the centralizer of G/R in the image of $\langle g \rangle$. We check condition (2): if $am = g^i$ for $a \in G$, $m \in M$, then $g^i \in G$ and therefore $g^i \in C_{\langle g \rangle}(G/R)$ by condition (2) for $G\langle g \rangle$. By induction we find $z \in G$ such that $H^z \cap \langle g \rangle \leq \langle g_0 \rangle M$; then $H^z \cap \langle g \rangle \leq \langle g_0 \rangle M \cap \langle g \rangle \leq C_{\langle g \rangle}(G/R) = \langle g_0 \rangle$, as required. Thus, we can assume that $G \leq MH$.

Suppose that $\langle g_0 \rangle = C_{\langle g \rangle}(G/R)$ contains some nontrivial Sylow q -subgroup $\langle g_q \rangle$ of $\langle g \rangle$. Then $\langle g \rangle = \langle g_1 \rangle \times \langle g_q \rangle$, where $\langle g_1 \rangle$ is (necessarily nontrivial) Hall q' -subgroup. We choose a minimal g_1 -invariant subgroup G_1 of G covering G/R . Then the subgroups $G_1^{g_q^j}$ are of similar nature and together generate G by minimality of G . Since $G \not\leq H$, we can assume without loss of generality that $G_1 \not\leq H$. The hypotheses of the proposition are satisfied for $G_1\langle g_1 \rangle$ and $H \cap G_1\langle g_1 \rangle$. Indeed, (1) is true by construction, (2) is inherited from the same condition for g , while (3) and (4) hold because the action of g_1 on G_1/R_1 is similar to the action of g on G/R . Since $G_1 \not\leq H \cap G_1\langle g_1 \rangle$, by induction we find $z \in G_1$ such that $(H \cap G_1\langle g_1 \rangle)^z \cap \langle g_1 \rangle \leq \langle g_0 \rangle$, whence $H^z \cap \langle g_1 \rangle \leq \langle g_0 \rangle$. Then, since $\langle g_q \rangle \leq \langle g_0 \rangle$ by assumption,

$$H^z \cap \langle g \rangle = (H^z \cap \langle g_1 \rangle) \times (H^z \cap \langle g_q \rangle) \leq \langle g_0 \rangle.$$

Thus, we can assume that every Sylow q -subgroup of $\langle g \rangle$ acts nontrivially on G/R .

Now suppose that $GH \neq G\langle g \rangle$; then $GH = G\langle g^s \rangle$ for some prime s . Note that $\langle g^s \rangle \geq \langle g_0 \rangle$ by the assumption at the end of the last paragraph. For a g^s -orbit on $\{S_i\}$, we choose a minimal g^s -invariant subgroup G_1 of G covering the product of the S_i in this g^s -orbit. Then the subgroups $G_1^{g^s}$ are of similar nature and $G = \langle G_1, G_1^{g^s}, \dots, G_1^{g^{s-1}} \rangle$ by minimality of G . Since $G \not\leq H$, we can assume without loss of generality that $G_1 \not\leq H$. Note that the centralizer of G_1R/R in $\langle g^s \rangle$ is equal to $\langle g_0 \rangle$ because the products of simple factors over g^s -orbits are permuted by g . The group $G_1\langle g^s \rangle$ satisfies the hypotheses of the proposition with $H_1 = H \cap G_1\langle g^s \rangle$ in place of H . Indeed, conditions (1), (3) are

satisfied by construction of G_1 , and conditions (2), (4) are obviously inherited from the same conditions for $\langle g \rangle$.

By induction we find $z \in G_1$ such that $H_1^z \cap \langle g^s \rangle \leq \langle g_0 \rangle$. Note that $G\langle g^s \rangle \cap \langle g \rangle = \langle g^s \rangle$, since $\langle g^s \rangle \geq \langle g_0 \rangle$ and $G \cap \langle g \rangle \leq \langle g_0 \rangle$ by condition (2). Then

$$\begin{aligned} H^z \cap \langle g \rangle &= H^z \cap G\langle g^s \rangle \cap \langle g \rangle = H^z \cap \langle g^s \rangle \\ &= H^z \cap G_1\langle g^s \rangle \cap \langle g^s \rangle = (H \cap G_1\langle g^s \rangle)^z \cap \langle g^s \rangle \\ &= H_1^z \cap \langle g^s \rangle \leq \langle g_0 \rangle \end{aligned}$$

Thus, we can assume that $GH = G\langle g \rangle$.

It follows that $M \cap H = 1$ by minimality of M . Indeed, this intersection is normal in MH because M is abelian, and $MH = G\langle g \rangle$ because $MH \geq GH = G\langle g \rangle$. We also have $M \not\leq H$, since $G \not\leq H$.

Let a_1, \dots, a_s be elements of prime orders in $\langle g \rangle$, one for each prime divisor of $|g|$. Suppose that for some i_0 we have $a_{i_0}^x \notin H$ for every $x \in G$, and let $|a_{i_0}| = q$. Let $\langle g_1 \rangle$ be the Hall q' -subgroup of $\langle g \rangle$. We can assume that g_1 acts nontrivially on G/R , for otherwise $\langle g_1 \rangle \leq \langle g_0 \rangle$, and then $H \cap \langle g \rangle \leq \langle g_0 \rangle$ and we are done. The products of simple factors of G/R over g_1 -orbits are transitively permuted by g . Hence g_1 acts nontrivially on each such product, and the centralizer of such a product in $\langle g_1 \rangle$ is equal to $\langle g_1 \rangle \cap \langle g_0 \rangle$. As above, since $G \not\leq H$, we can find a g_1 -orbit on $\{S_i\}$ such that H does not contain a minimal g_1 -invariant subgroup G_1 of G covering the product of the S_i in this g_1 -orbit. Then the hypotheses of the proposition hold for G_1 , g_1 , and $H_1 = H \cap G_1\langle g_1 \rangle$ in place of G , g , and H . Indeed, conditions (1), (3) hold by construction, condition (4) is inherited from the same condition for g , and condition (2) follows from the same condition for g because an element in $\langle g_1 \rangle$ centralizing G_1R/R also centralizes G/R . By induction we find $z \in G_1$ such that $H_1^z \cap \langle g_1 \rangle \leq \langle g_0 \rangle$, and therefore also

$$H^z \cap \langle g_1 \rangle = H^z \cap G_1\langle g_1 \rangle \cap \langle g_1 \rangle = (H \cap G_1\langle g_1 \rangle)^z \cap \langle g_1 \rangle = H_1^z \cap \langle g_1 \rangle \leq \langle g_0 \rangle.$$

Since $a_{i_0}^{z^{-1}} \notin H$ by our assumption, we obtain $H^z \cap \langle g \rangle \leq \langle g_0 \rangle$. Thus, we can assume that for every a_i there is $x \in G$ such that $a_i^x \in H$. Since $G \leq MH$, for every i we can choose $m_i \in M$ such that $a_i^{m_i} \in H$.

We set

$$M_i = \{m \in M \mid a_i^m \in H\}.$$

Since M is abelian, a direct calculation shows that $M_i = m_i C_M(a_i)$. Indeed, if $a_i^m \in H$, then $a_i^m (a_i^{-1})^{m_i} \in M \cap H = 1$, whence $a_i^{mm_i^{-1}} a_i^{-1} = 1$, which means that $m \in m_i C_M(a_i)$. Thus, $M_i \subseteq m_i C_M(a_i)$. The reverse inclusion is obvious: $a_i^{mm_i} = a_i^{m_i} \in H$ for any $m \in C_M(a_i)$.

If there is $z \in M \setminus \bigcup M_i$, then $a_i^z \notin H$ for all i , so that $H^{z^{-1}} \cap \langle g \rangle = 1$ and the proof is complete. Therefore we can assume that $M = \bigcup M_i$.

Suppose that $M = M_{i_0}$ for some i_0 . Then $M = C_M(a_{i_0})$, which implies that $a_{i_0} \in H$ and, moreover, $a_{i_0}^x \in H$ for any $x \in G\langle g \rangle$, since $G \leq MH$ and g centralizes a_{i_0} . In other words,

$$a_{i_0} \in K := \bigcap_{x \in G\langle g \rangle} H^x,$$

where K is a normal subgroup of $G\langle g \rangle$. It now follows that $a_{i_0} \in \langle g_0 \rangle$: otherwise, $[G/R, a_{i_0}] = G/R$ and $[G, a_{i_0}] \leq [G, K] \leq K$, so that $K \cap G$ covers G/R contrary to minimality of G , since $G \not\leq H \geq K$.

First suppose that $K_0 = K \cap R \neq 1$. Then the hypotheses of the proposition hold for the images in $G\langle g \rangle/K_0$, which we denote by tilde. Indeed, $\tilde{G}/\tilde{R} \cong G/R$ and $C_{\langle \tilde{g} \rangle}(\tilde{G}/\tilde{R}) = \langle \tilde{g}_0 \rangle$.

Conditions (1), (3), (4) obviously hold since $K_0 \leq R$. We check condition (2): if $ak = g^i$ for $a \in G$, $k \in K_0$, then $g^i \in \langle g_0 \rangle$ by condition (2) for $G\langle g \rangle$. Also, $\tilde{G} \not\leq \tilde{H}$ since $K_0 \leq G \cap H$. By induction we find $\tilde{z} \in \tilde{G}$ such that $\tilde{H}^{\tilde{z}} \cap \langle \tilde{g} \rangle \leq \langle \tilde{g}_0 \rangle$. Then for a pre-image $z \in G$ we have

$$H^z \cap \langle g \rangle \leq \langle g_0 \rangle K_0 \cap \langle g \rangle \leq C_{\langle g \rangle}(G/R) = \langle g_0 \rangle,$$

as required.

Thus we can assume that $K \cap R = 1$. Then $[a_{i_0}, G] \leq K \cap R = 1$, so a_{i_0} is central in $G\langle g \rangle$. We are going to apply induction to $G\langle g \rangle / \langle a_{i_0} \rangle$, where the images are denoted by tilde. Clearly, $\tilde{G}/\tilde{R} \cong G/R$. We claim that $C_{\langle \tilde{g} \rangle}(\tilde{G}/\tilde{R}) = \langle \tilde{g}_0 \rangle$. Indeed, if $[G, g^i] \leq R\langle a_{i_0} \rangle$, then $[[G, g^i], g^i] \leq [R\langle a_{i_0} \rangle, g^i] \leq R$, whence $[G, g^i] \leq R$, since then the image of g^i in $G\langle g \rangle/R$ belongs to the Fitting subgroup by Baer's theorem. We now verify the other hypotheses of the proposition. Condition (1): if $\tilde{G}_1 \leq \tilde{G}$ for a g -invariant subgroup \tilde{G}_1 covering \tilde{G}/\tilde{R} , then for the full inverse image we have $G_1 \leq G\langle a_{i_0} \rangle$. Then $[G_1, G_1] \leq [G\langle a_{i_0} \rangle, G\langle a_{i_0} \rangle] \leq G$. Since $[G_1, G_1]$ also covers G/R , by minimality of G we must have $[G_1, G_1] = G$, whence $\tilde{G}_1 = \tilde{G}$. Condition (2): if $b = g^i a_{i_0}^j$ for $b \in G$, then $g^i a_{i_0}^j \in \langle g_0 \rangle$ by condition (2) for $G\langle g \rangle$, whence $g^i \in \langle g_0 \rangle$, since $a_{i_0} \in \langle g_0 \rangle$. Conditions (3) and (4) obviously follow from the same conditions for $G\langle g \rangle$. Finally, $\tilde{G} \not\leq \tilde{H}$, since otherwise $G \leq H\langle a_{i_0} \rangle = H$, contrary to the hypothesis. By induction we find $z \in G$ such that $H^z \cap \langle g \rangle \leq \langle g_0 \rangle \langle a_{i_0} \rangle = \langle g_0 \rangle$.

Thus, we can assume that $M_i \neq M$ for every i .

Consider the subgroup $A = \langle a_1^{m_1}, \dots, a_s^{m_s} \rangle \leq H$. Since $H \cap M = 1$, we have $A \cong AM/M \cong \langle a_1, \dots, a_s \rangle$, which is a cyclic group (a subgroup of $\langle g \rangle$). The subgroup M is an elementary abelian q -group for some prime q . Let B be a Hall q' -subgroup of A (possibly, $B = A$). In the semidirect product $MA = M\langle a_1, \dots, a_s \rangle$, the Hall q' -subgroup of $\langle a_1, \dots, a_s \rangle$, which is also a Hall q' -subgroup of MA , is conjugate to B by an element $y \in M$. Since $B \leq H$, this means that for every i such that $|a_i| \neq q$ we can replace all those elements m_i by this element $y \in m_i C_M(a_i)$. Without loss of generality, suppose that all the elements a_2, \dots, a_s are of order coprime to q , while $|a_1|$ may be equal to q , or not. If $|a_1| \neq q$, then

$$M = yC_M(a_1) \cup yC_M(a_2) \cup \dots \cup yC_M(a_s),$$

whence, after multiplying by y^{-1} ,

$$M = C_M(a_1) \cup C_M(a_2) \cup \dots \cup C_M(a_s). \quad (4.1)$$

If $|a_1| = q$, then

$$M = m_1 C_M(a_1) \cup yC_M(a_2) \cup \dots \cup yC_M(a_s),$$

whence, after multiplying by y^{-1} ,

$$M = y^{-1}m_1 C_M(a_1) \cup C_M(a_2) \cup \dots \cup C_M(a_s). \quad (4.2)$$

Suppose that $y^{-1}m_1 C_M(a_1) \neq C_M(a_1)$. In this case, let $b = a_2 \dots a_s$. If b does not normalize the coset $y^{-1}m_1 C_M(a_1)$, then (4.2) implies

$$(y^{-1}m_1 C_M(a_1))^b \subseteq C_M(a_2) \cup \dots \cup C_M(a_s)$$

and then also

$$y^{-1}m_1 C_M(a_1) \subseteq C_M(a_2) \cup \dots \cup C_M(a_s),$$

since the right-hand side is b -invariant. Then

$$M = C_M(a_2) \cup \dots \cup C_M(a_s). \quad (4.3)$$

If b normalizes the coset $y^{-1}m_1 C_M(a_1)$, then, since the action of b on M is coprime, it follows that $y^{-1}m_1 C_M(a_1)$ contains an element $m_0 \in C_M(b) = \bigcap_{i=2}^s C_M(a_i)$. For any $x \in$

$m_0 C_M(a_1) = y^{-1} m_1 C_M(a_1) \neq C_M(a_1)$ the element $m_0 x$ is not in the coset $y^{-1} m_1 C_M(a_1)$ and therefore $m_0 x \in C_M(a_i)$ for some $i \geq 2$ by (4.2). Then $x \in m_0^{-1} C_M(a_i) = C_M(a_i)$, so that again

$$y^{-1} m_1 C_M(a_1) \subseteq C_M(a_2) \cup \cdots \cup C_M(a_s)$$

and by (4.2)

$$M = C_M(a_2) \cup \cdots \cup C_M(a_s). \quad (4.4)$$

As a result of (4.1), (4.3), (4.4), in all cases,

$$M = C_M(a_1) \cup C_M(a_2) \cup \cdots \cup C_M(a_s).$$

Note also that the automorphism $a_1 a_2 \cdots a_s$ acts faithfully on M , as $M \neq C_M(a_i)$ for every i . This situation is known to be impossible due to the following well-known lemma, which we prove here for completeness.

Lemma 4.6. *If α is an automorphism of an elementary abelian q -group V , then there is an element $v \in V$ such that $C_{\langle \alpha \rangle}(v) = 1$.*

Proof. Induction on $|V|$. If $|V| = q$, then every element in $\langle \alpha \rangle$ acts without non-trivial fixed points, and the result follows. In the general case, if an element $\beta \in \langle \alpha \rangle$ of prime order $r \neq q$ has non-trivial fixed points, then $V = C_V(\beta) \times [V, \beta]$ by Maschke's theorem, and both factors are α -invariant and have smaller order than V . By induction, there are $v_1 \in C_V(\beta)$ and $v_2 \in [V, \beta]$ such that $C_{\langle \alpha \rangle}(v_1) = C_{\langle \alpha \rangle}(C_V(\beta))$ and $C_{\langle \alpha \rangle}(v_2) = C_{\langle \alpha \rangle}([V, \beta])$. Then $C_{\langle \alpha \rangle}(v_1 + v_2) = C_{\langle \alpha \rangle}(C_V(\beta)) \cap C_{\langle \alpha \rangle}([V, \beta]) = C_{\langle \alpha \rangle}(V) = 1$.

Thus, we can assume that all elements of $\langle \alpha \rangle$ of order coprime to q act on V without nontrivial fixed points. It remains to choose an element of V outside the centralizer of an element of $\langle \alpha \rangle$ of order q (or any non-trivial element of V if $q \nmid |\alpha|$). \square

The proof of Proposition 4.5 is complete. \square

5. NONSOLUBLE LENGTH

In this section we prove Theorem 1.3. First we introduce some notation. Consider the 'upper nonsoluble series' of G , which by definition starts from the soluble radical $R_0 = R(G)$ and the full inverse image L_1 of $F^*(G/R_0)$. Then by induction R_j is the full inverse image of the soluble radical of G/L_j , and L_{j+1} the full inverse image of $F^*(G/R_j)$. It is easy to see that the nonsoluble length $\lambda(G)$ is equal to the first positive integer l such that $R_l = G$. In the normal series

$$1 = L_0 \leq R_0 < L_1 \leq R_1 < \cdots \leq R_l = G \quad (5.1)$$

each quotient $U_i = L_i/R_{i-1}$ is a (nontrivial) direct product of nonabelian simple groups, and each quotient R_i/L_i is soluble (possibly trivial). By the well-known properties of the generalized Fitting subgroup, if we write one of those nonsoluble quotients as a direct product $U_j = S_1 \times \cdots \times S_v$ of nonabelian simple groups S_i , then the set of these factors S_i is uniquely determined as the set of subnormal simple subgroups of G/R_{j-1} . Acting by conjugation the group G permutes these subnormal factors; for brevity we simply speak of orbits of elements of G on U_j meaning orbits in this permutational action. The stabilizer of a point S_i can also be denoted as the normalizer $N_G(S_i)$ of the section S_i .

The subgroup L_j/R_{j-1} contains its centralizer in G/R_{j-1} . Let K_j be the kernel of the permutational action of G on $\{S_1, \dots, S_v\}$. Clearly, $L_j \leq K_j$. The quotient K_j/L_j is soluble by the Schreier conjecture. Therefore, $K_j \leq R_j$. We shall routinely use these facts without special references.

Let g be an element of a finite group G , and let $\{S_1, \dots, S_r\}$ be a g -orbit in U_i . We say that the orbit is *pure* if the order of the automorphism of $S = S_1 \cdots S_r$ induced by g acting by conjugation is equal to r , which is the order of the permutation induced by g on this orbit; in other words, if the stabilizer of a point in $\langle g \rangle$ acts trivially on S : $N_{\langle g \rangle}(S_1) = C_{\langle g \rangle}(S)$.

We now prove two key technical propositions. It is convenient to introduce the following hypothesis.

Hypothesis 5.1. *Let g be an element of a finite group G , and let $\{S_1, \dots, S_r\}$ be a g -orbit in a section $U_i = L_i/R_{i-1}$ of the series (5.1) with $i \geq 2$. Suppose that $\langle g \rangle \cap L_i = 1$, and that g acting by conjugation induces a nontrivial automorphism \bar{g} of $S = S_1 \cdots S_r$. Let $t = |\bar{g}|$, so that $\langle g^t \rangle$ is the centralizer of S in $\langle g \rangle$ and $\langle \bar{g} \rangle = \langle g \rangle / \langle g^t \rangle$.*

Furthermore, let \hat{S} be a minimal by inclusion g -invariant subgroup of G such that $\hat{S}R_{i-1}/R_{i-1} = S$. If in addition $\{S_1, \dots, S_r\}$ is a pure g -orbit, then we choose \hat{S} to be also contained in $E_n(g)$, which is possible by Lemma 3.9. Let \hat{S}_1 be a minimal by inclusion subgroup of \hat{S} such that $\hat{S}_1 R_{i-1}/R_{i-1} = S_1$.

The first of the technical propositions provides a passage from a pure orbit of g in U_i to an orbit of at least the same length in the preceding section U_{i-1} .

Proposition 5.2. *Assume Hypothesis 5.1. If $\{S_1, \dots, S_r\}$ is a pure g -orbit, then U_{i-1} contains a g -orbit $\{T_1, \dots, T_l\}$ of length divisible by $|\bar{g}|$ and for some choice of \hat{S}_1 there is an element $x_0 \in \hat{S}_1 \setminus R_{i-1}$ that does not belong to $N_G(T_1)$.*

Proof. Consider the permutational action of the group $\hat{S}\langle g \rangle$ on the set of simple factors of U_{i-1} . Since the kernel of this action is contained in R_{i-1} , for an element $x_0 \in \hat{S}_1 \setminus R_{i-1}$ there is an orbit $\{T_1, \dots, T_s\}$ such that the kernel K of the restriction to this orbit does not contain x_0 . In particular, $\hat{S} \not\leq K$. Then K cannot cover S by minimality of \hat{S} . Since K is a normal subgroup of $\hat{S}\langle g \rangle$, it follows that $K \leq (\hat{S} \cap R_{i-1})\langle g^t \rangle$ (where, recall, $\langle g^t \rangle$ is the centralizer of S in $\langle g \rangle$). In particular, $K \leq C_{\hat{S}\langle g \rangle}(S)$.

Let $W = \hat{S}K/K$ and let R be the soluble radical of W ; then $W/R \cong S$ since $L_{j-1} \leq K$. Let \tilde{g} be the image of g in $\hat{S}\langle g \rangle/K$. We claim that $C_{\langle \tilde{g} \rangle}(S) = \langle \tilde{g}^t \rangle$. Indeed, if $[\hat{S}, g^i] \leq (R_{i-1} \cap \hat{S})K \leq R_{i-1}\langle g^t \rangle$, then $[[\hat{S}, g^i], g^i] \leq R_{i-1}$, whence $[\hat{S}, g^i] \leq R_{i-1}$, since then the image of g^i in $\hat{S}\langle g \rangle/(R_{i-1} \cap \hat{S})$ belongs to the Fitting subgroup by Baer's theorem.

We consider the action of $W\langle \tilde{g} \rangle$ on the orbit $\{T_1, \dots, T_s\}$. Let \tilde{H} be the stabilizer of T_1 in $W\langle \tilde{g} \rangle$. Then $W \not\leq \tilde{H}$, since $\hat{S} \not\leq K$. We now show that the triple W, \tilde{g}, \tilde{H} satisfies the other hypotheses of Proposition 4.5. Condition (1): if $\tilde{M} \leq W$ for a g -invariant subgroup \tilde{M} covering S , then for the full inverse image M we have $M \leq \hat{S}K \leq \hat{S}\langle g^t \rangle$. Then $[M, M] \leq [\hat{S}\langle g^t \rangle, \hat{S}\langle g^t \rangle] \leq \hat{S}$. Since $[M, M]$ also covers S , by minimality of \hat{S} we must have $[M, M] = \hat{S}$, whence $\tilde{M} = W$. Condition (2): if $ak = g^i$ for $a \in \hat{S}$, $k \in K$, then $b = g^i g^{tj}$ for some $b \in \hat{S}$ and some j , because $K \leq (\hat{S} \cap R_{i-1})\langle g^t \rangle$. Since $\langle g \rangle \cap \hat{S} = 1$ by Hypothesis 5.1, we obtain $b = g^i g^{tj} = 1$, so that $g^i \in \langle g^t \rangle$ and $\tilde{g}^i \in C_{\langle \tilde{g} \rangle}(S)$. Conditions (3) and (4) clearly hold.

By Proposition 4.5 there is $\tilde{w} \in W$ such that

$$\tilde{H}^{\tilde{w}} \cap \langle \tilde{g} \rangle \leq \langle \tilde{g}^t \rangle.$$

Let H be the full inverse image of \tilde{H} in $\hat{S}\langle g \rangle$, and w some pre-image of \tilde{w} . Then

$$H^w \cap \langle g \rangle \leq \langle g^t \rangle K \cap \langle g \rangle \leq C_{\hat{S}\langle g \rangle}(S) \cap \langle g \rangle = \langle g^t \rangle. \quad (5.2)$$

Since H is the stabilizer of T_1 in $\hat{S}\langle g \rangle$, this means that the g -orbit in U_{i-1} containing T_1^w has length divisible by $t = |\bar{g}|$.

Recall that $x_0 \in \hat{S}_1 \setminus R_{i-1}$ and $x_0 \notin K$. There is $y \in \hat{S}\langle g \rangle$ such that $x_0 \notin H^{wy}$. Since $y = g^j s$ for some integer j and some $s \in \hat{S}$, we have $x_0 \notin H^{wy} = H^{wg^j s}$, so that $x_0^{s^{-1}} \notin H^{wg^j}$. Thus the element $x_0^{s^{-1}}$, which belongs to $\hat{S}_1^{s^{-1}} \setminus R_{i-1}$, moves $T_1^{wg^j}$. Note that $\hat{S}_1^{s^{-1}}$ is also a minimal subgroup of \hat{S} covering S_1 . Clearly, the g -orbit containing $T_1^{wg^j}$ has the same length as the g -orbit containing T_1^w , and this length divides $|\bar{g}|$ by (5.2). It remains to rename $T_1^{wg^j}$ by T_1 , $\hat{S}_1^{s^{-1}}$ by \hat{S}_1 , and $x_0^{s^{-1}}$ by x_0 . The proposition is proved. \square

The next technical proposition provides a passage from a non-pure orbit of g in U_i to an orbit in U_{i-1} that is strictly greater with respect to the following ordering. Namely, on the set of positive integers we introduce the lexicographical order with respect to the exponents of primes in the canonical prime-power decomposition: if

$$a = 2^{k_2} \cdot 3^{k_3} \cdot 5^{k_5} \cdot 7^{k_7} \dots \quad \text{and} \quad b = 2^{l_2} \cdot 3^{l_3} \cdot 5^{l_5} \cdot 7^{l_7} \dots,$$

then by definition $a \prec b$ if, for some prime p , we have $k_q = l_q$ for all primes $q < p$ and $k_p < l_p$. Clearly, if u is divisible by v , then $v \prec u$.

Proposition 5.3. *Assume Hypothesis 5.1. If $\{S_1, \dots, S_r\}$ is a non-pure g -orbit, then U_{i-1} contains a g -orbit $\{T_1, \dots, T_l\}$ of length strictly greater than r with respect to the order \prec .*

Proof. Let $|\bar{g}| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots$ for primes $p_1 < p_2 < \dots$ and positive integers α_i . Let \bar{g}_i be a generator of the Sylow p_i -subgroup of $\langle \bar{g} \rangle$, and let g_i be an inverse image of \bar{g}_i in the Sylow p_i -subgroup of $\langle g \rangle$.

Since the orbit is non-pure, the stabilizer of a point S_1 in the permutational action of \bar{g} on $\{S_1, \dots, S_r\}$ is nontrivial. Let the order of this stabilizer be $p_1^{\beta_1} \cdot p_2^{\beta_2} \dots$ for the same primes p_i and non-negative integers β_i , not all of which are zero. Let j be the smallest index such that $\beta_j \geq 1$. Consider the element

$$f = g_1 \cdots g_{j-1} \cdot g_j$$

(which generates the Hall $\{p_1, \dots, p_j\}$ -subgroup of $\langle g \rangle$). Here, the case $j = 1$ is not excluded, when $f = g_1$. Let \bar{f} be the image of f in $\langle g \rangle / \langle g^t \rangle$, so that

$$|\bar{f}| = p_1^{\alpha_1} \cdots p_j^{\alpha_j}.$$

Let $t_0 = |\bar{f}|$ for brevity.

Consider an orbit of f in $\{S_1, \dots, S_r\}$, which we denote by $\{V_1, \dots, V_v\}$. Let $V = V_1 \cdots V_v$, and let \hat{V} be a minimal f -invariant subgroup of \hat{S} such that $\hat{V}R_{i-1}/R_{i-1} = V$. Let \check{f} be the automorphism of V induced by f . Since the products of simple factors over f -orbits are permuted by g , the centralizer of V in $\langle f \rangle$ is equal to $\langle f \rangle \cap \langle g^t \rangle = \langle f^{t_0} \rangle$ (recall that $\langle g^t \rangle$ is the centralizer of S in $\langle g \rangle$). Hence,

$$|\check{f}| = |\bar{f}| = p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}} \cdot p_j^{\alpha_j}.$$

Note that by construction the stabilizer of a point V_1 in $\langle \check{f} \rangle$ is a p_j -subgroup.

Consider the permutational action of the group $\hat{V}\langle \check{f} \rangle$ on the set of simple factors of U_{i-1} . Since the kernel of this action is contained in R_{i-1} , we can choose an orbit $\{T_1, \dots, T_l\}$ for which the kernel K of the restriction to this orbit does not contain some element $x \in \hat{V}_1 \setminus R_{i-1}$. Then K cannot cover V by minimality of \hat{V} . Since K is a normal subgroup of $\hat{V}\langle \check{f} \rangle$, it follows that $K \leq (\hat{V} \cap R_{i-1})\langle f^{t_0} \rangle$. In particular, $K \leq C_{\hat{V}\langle \check{f} \rangle}(V)$.

Let $W = \hat{V}K/K$ and let R be the soluble radical of W ; then $W/R \cong V$, since $L_{j-1} \leq K$. Let \tilde{f} be the image of f in $\hat{V}\langle f \rangle/K$. We claim that $C_{\langle \tilde{f} \rangle}(V) = \langle \tilde{f}^{t_0} \rangle$. Indeed, if $[\hat{V}, f^i] \leq (R_{i-1} \cap \hat{V})K \leq R_{i-1}\langle g^t \rangle$, then $[[\hat{V}, f^i], f^i] \leq R_{i-1}$, whence $[\hat{V}, f^i] \leq R_{i-1}$, since then the image of f^i in $\hat{V}\langle f \rangle/(R_{i-1} \cap \hat{V})$ belongs to the Fitting subgroup by Baer's theorem.

We consider the action of $W\langle \tilde{f} \rangle$ on the orbit $\{T_1, \dots, T_l\}$. Let \tilde{H} be the stabilizer of T_1 in $W\langle \tilde{f} \rangle$. Then $W \not\leq \tilde{H}$, since $\hat{V} \not\leq K$.

We now show that the triple W, \tilde{f}, \tilde{H} satisfies the other hypotheses of Proposition 4.5 in the role of G, g, H . Condition (1): if $\tilde{M} \leq W$ for an \tilde{f} -invariant subgroup \tilde{M} covering V , then for the full inverse image M we have $M \leq \hat{V}K \leq \hat{S}\langle f^{t_0} \rangle$. Then $[M, M] \leq [\hat{V}\langle f^{t_0} \rangle, \hat{V}\langle f^{t_0} \rangle] \leq \hat{V}$. Since $[M, M]$ also covers V , by minimality of \hat{V} we must have $[M, M] = \hat{V}$, whence $\tilde{M} = W$. Condition (2): if $ak = f^i$ for $a \in \hat{V}$, $k \in K$, then $b = f^i f^{t_0 j}$ for some $b \in \hat{V}$ and some j , because $K \leq (\hat{V} \cap R_{i-1})\langle f^{t_0} \rangle$. Since $\langle f \rangle \cap \hat{V} = 1$ by Hypothesis 5.1, we obtain $b = f^i f^{t_0 j} = 1$, so that $f^i \in \langle f^{t_0} \rangle$ and $\tilde{f}^i \in C_{\langle \tilde{f} \rangle}(V)$. Conditions (3) and (4) clearly hold by construction.

By Proposition 4.5 there is $\tilde{w} \in W$ such that

$$\tilde{H}^{\tilde{w}} \cap \langle \tilde{f} \rangle \leq \langle \tilde{f}^{t_0} \rangle.$$

Let H be the full inverse image of \tilde{H} in $\hat{V}\langle f \rangle$, and w some pre-image of \tilde{w} . Then

$$H^w \cap \langle f \rangle \leq \langle f^{t_0} \rangle K \cap \langle f \rangle \leq C_{\hat{V}\langle f \rangle}(V) \cap \langle f \rangle = \langle f^{t_0} \rangle.$$

Since H is the stabilizer of T_1 in $\hat{V}\langle f \rangle$, this means that the f -orbit in U_{i-1} containing T_1^w has length divisible by

$$|\tilde{f}| = p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}} \cdot p_j^{\alpha_j}.$$

Then the g -orbit containing T_1^w also has length divisible by

$$p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}} \cdot p_j^{\alpha_j}.$$

This number is strictly greater with respect to our lexicographical order \prec than the length of the original orbit

$$p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}} \cdot p_j^{\alpha_j - \beta_j} \cdots$$

by the choice of j . It remains to rename T_1^w by T_1 . The proposition is proved. \square

Theorem 1.3 will follow by induction from the following proposition. Recall that K_i is the kernel of the permutational action of G on the set of simple factors of $U_i = L_i/R_{i-1}$.

Proposition 5.4. *Let g be an element of a finite group G whose order $|g|$ is equal to the product of m primes counting multiplicities. Suppose that $\langle g \rangle \cap K_{ms} = 1$ for some positive integer s . Then for any positive integer n the nonsoluble length of $E_n(g)$ is at least s .*

Proof. Consider the ‘upper nonsoluble series’ for $E_n(g)$ constructed in the same way as (5.1) was constructed for G , with its terms denoted by

$$1 = \lambda_0 \leq \rho_0 < \lambda_1 \leq \rho_1 < \cdots \leq \rho_e = E_n(g), \quad (5.3)$$

where $e = \lambda(E_n(g))$ is the nonsoluble length of $E_n(g)$. Namely, $\rho_0 = R(E_n(g))$ is the soluble radical, λ_1 is the full inverse image of $F^*(E_n(g)/\rho_0)$, and by induction ρ_j is the full inverse image of the soluble radical of $E_n(g)/\lambda_j$, and λ_{j+1} the full inverse image of $F^*(E_n(g)/\rho_j)$. The quotients λ_i/ρ_{i-1} are (nontrivial) direct products of nonabelian simple groups, and the quotients ρ_i/λ_i (possibly trivial) are soluble. Thus, e is the first positive integer such that $\rho_e = E_n(g)$. Our task is to show that $e \geq s$.

Since $\langle g \rangle \cap K_{ms} = 1$, the element g has at least one nontrivial orbit on the set of simple factors of $U_{ms} = L_{ms}/R_{ms-1}$, say, $\{S_1, \dots, S_r\}$. If the orbit is pure, then we apply Proposition 5.2 to this orbit. If the orbit is not pure, then we apply Proposition 5.3. Then we apply the same procedure to the orbit $\{T_1, \dots, T_l\}$ in U_{ms-1} thus obtained: this orbit takes the role of the orbit $\{S_1, \dots, S_r\}$ in Proposition 5.2 or 5.3 depending on whether it is pure or not. We proceed with constructing this sequence of orbits, descending over the sections U_i making $ms - 1$ such steps. If we make such a step from a pure orbit by Proposition 5.2, then the length of the new orbit is divisible by the length of the old orbit and therefore does not decrease with respect to the order \prec . If we make such a step from a non-pure orbit by Proposition 5.3, then the length of the new orbit is strictly greater than the length of the old orbit with respect to the order \prec .

In the sequence of orbits thus constructed, some orbits may be pure, some not. We can visualize this fact as a sequence of P s (for pure) and N s (for non-pure), like

$$P \rightarrow N \rightarrow P \rightarrow P \rightarrow P \rightarrow N \rightarrow N \rightarrow P \rightarrow N \rightarrow P \rightarrow \dots$$

Importantly, in every passage of types $N \rightarrow N$ or $N \rightarrow P$ the length of the orbit strictly increases with respect to the order \prec , while at passages of types $P \rightarrow N$ or $P \rightarrow P$ the length of the orbit does not decrease with respect to the order \prec . Therefore there can be at most $m - 1$ passages of type $N \rightarrow N$ or $N \rightarrow P$, that is, at most $m - 1$ occurrences of N in this sequence. As a result, if the length of the sequence is at least $(s - 1)m + (m - 1) + 1 = ms$, then it will necessarily contain a subsequence

$$P \rightarrow P \rightarrow \dots \rightarrow P$$

of s pure orbits with $s - 1$ consecutive passages $P \rightarrow P$. Let $i = t, t - 1, \dots, t - s + 1$ be the indices of the corresponding sections U_i .

Recall that if a g -orbit $\{S_1, \dots, S_r\}$ in U_i is pure, then by Lemma 3.9 the subgroup $S = S_1 \times \dots \times S_r$ is contained in the image of $E_n(g)$ in G/R_{i-1} (since this image is obviously equal to the analogous subgroup $E_n(g)$ constructed for G/R_{i-1}).

The idea is to use each of these s consecutive pure orbits in $U_t, U_{t-1}, \dots, U_{t-s+1}$ to ‘mark’ a nonsoluble factor of the series (5.3) and prove that the factor ‘marked’ by the pure orbit in U_{i-1} is necessarily ‘lower’ in (5.3) than the factor marked by the pure orbit in U_i , for every $i = t, t - 1, \dots, t - s + 2$. Then the series (5.3) must realize nonsoluble length at least s .

Thus, let $\{S_1, \dots, S_r\}$ be a pure g -orbit in U_i , and $\{T_1, \dots, T_l\}$ the pure g -orbit in U_{i-1} obtained by Proposition 5.2. Recall that in accordance with Hypothesis 5.1, \hat{S} is a minimal by inclusion g -invariant subgroup of $E_n(g)$ such that $S = \hat{S}R_{i-1}/R_{i-1}$, and \hat{S}_1 is a minimal by inclusion subgroup of \hat{S} such that $\hat{S}_1 R_{i-1}/R_{i-1} = S_1$. Note that since S_1 is nonabelian simple, \hat{S}_1 has no nontrivial soluble homomorphic images:

$$\hat{S}_1 = [\hat{S}_1, \hat{S}_1]. \quad (5.4)$$

Recall also that by Proposition 5.2 we have an element $x_0 \in \hat{S}_1 \setminus R_{i-1}$ such that

$$T_1^{x_0} \neq T_1. \quad (5.5)$$

Consider the image of the series (5.3) in $E_n(g)R_{i-1}/R_{i-1}$. Since $\hat{S}_1 R_{i-1}/R_{i-1} \cong S_1$ is a subnormal nonabelian simple subgroup of $E_n(g)R_{i-1}/R_{i-1}$, we obviously have a well-defined index j such that

$$\hat{S}_1 \leq \lambda_j R_{i-1} \quad \text{and} \quad \hat{S}_1 \not\leq \rho_{j-1} R_{i-1}. \quad (5.6)$$

Note that then also

$$\hat{S}_1 \leq \lambda_j(R_{i-1} \cap E_n(g)). \quad (5.7)$$

It is also clear that the index j depends only on S_1 (that is, it is independent of the choice of \hat{S}_1 such that $\hat{S}_1 R_{i-1}/R_{i-1} = S_1$).

Since $\{T_1, \dots, T_l\}$ is also a pure g -orbit in U_{i-1} , the product $T = T_1 \cdots T_l$ is also covered by $E_n(g)$ by Lemma 3.9. If \hat{T}_1 is any subgroup of $E_n(g)$ such that $\hat{T}_1 R_{i-2}/R_{i-2} = T_1$, then, again, there is a well-defined index u depending only on T_1 such that

$$\hat{T}_1 \leq \lambda_u(R_{i-2} \cap E_n(g)) \quad \text{and} \quad \hat{T}_1 \not\leq \rho_{u-1} R_{i-2}. \quad (5.8)$$

Lemma 5.5. *Under the above hypotheses, $j > u$.*

Proof. We argue by contradiction. Suppose that $j \leq u$. Then (5.7) implies that $\hat{S}_1 \leq \lambda_u(R_{i-1} \cap E_n(g))$. The image of λ_u/ρ_{u-1} in $E_n(g)/(R_{i-2} \cap E_n(g))$ is a direct product of nonabelian simple groups, one of which is $\bar{T}_1 = \hat{T}_1/(\hat{T}_1 \cap R_{i-2}) \cong T_1$ by (5.8). Acting by conjugation the group $E_n(g)$ permutes these factors. Consider the permutational action of $E_n(g)$ on the orbit containing \bar{T}_1 . Clearly, λ_u is contained in the kernel of this action. The subgroup $L_{i-1} \cap E_n(g)$ normalizes \hat{T}_1 modulo $R_{i-2} \cap E_n(g)$ (since L_{i-1} normalizes \hat{T}_1 modulo R_{i-2}). As a normal subgroup of $E_n(g)$, then $L_{i-1} \cap E_n(g)$ is contained in the kernel of that action on the orbit containing \bar{T}_1 . Since R_{i-1}/L_{i-1} is soluble, we obtain that the image of $\hat{S}_1 \leq \lambda_u(R_{i-1} \cap E_n(g))$ in this action is soluble. But $\hat{S}_1 = [\hat{S}_1, \hat{S}_1]$ by the minimality of \hat{S}_1 , as noted in (5.4). Therefore this image must actually be trivial; in particular,

$$\hat{T}_1^{x_0} \equiv \hat{T}_1 \pmod{\rho_{u-1}(R_{i-2} \cap E_n(g))}. \quad (5.9)$$

On the other hand, since $T_1^{x_0} \neq T_1$ by (5.5), it follows that $[\hat{T}_1, \hat{T}_1^{x_0}] \leq R_{i-2} \cap E_n(g)$. Together with (5.9) this implies

$$\bar{T}_1 = [\bar{T}_1, \bar{T}_1] \equiv 1 \pmod{\rho_{u-1}(R_{i-2} \cap E_n(g))},$$

contrary to (5.8). \square

We now finish the proof of Proposition 5.4. Each pure orbit $\{S_1, \dots, S_r\}$ in U_i for $i = t, \dots, t-s+1$ in our sequence constructed by successive application of Propositions 5.2 or 5.3 marks a nonsoluble quotient λ_j/ρ_{j-1} of the series (5.3) in the sense of (5.6). By Lemma 5.5 the next pure orbit in U_{i-1} marks a strictly lower section. Therefore there must be at least s different nonsoluble sections in (5.3), since their indices must be strictly descending as we go over the s consecutive pure orbits. As a result, the nonsoluble length of $E_n(g)$ is at least s . \square

Proof of Theorem 1.3. Recall that we have $\lambda(E_n(g)) = k$ and $|g|$ is a product of m primes counting multiplicities; we need to show that $g \in R_{(k+1)m(m+1)/2}(G)$. By Proposition 5.4 we have

$$\langle g \rangle \cap R_{(k+1)m} \geq \langle g \rangle \cap K_{(k+1)m} \neq 1.$$

The similar subgroup $E_n(\bar{g})$ constructed for the image \bar{g} of g in $\bar{G} = G/R_{(k+1)m}$ is clearly the image of $E_n(g)$ and therefore its nonsoluble length is at most k . Then $\bar{g} \in R_{(k+1)(m-1)m/2}(\bar{G})$ by induction on m . The result follows, since $R_{(k+1)(m-1)m/2}(\bar{G})$ is the image of $R_{(k+1)m(m+1)/2}$ in $G/R_{(k+1)m}$. \square

6. GENERALIZED FITTING HEIGHT

Proof of Theorem 1.2. Recall that g is an element of a finite group G whose order $|g|$ is equal to the product of m primes counting multiplicities, and the generalized Fitting height of $E_n(g)$ is equal to k . We need to show g belongs to $F_{((k+1)m(m+1)+2)(k+3)/2}^*(G)$.

As in the proof of Theorem 1.1, consider the subnormal g -invariant subgroups

$$[\dots[[G, g], g], \dots, g],$$

formed by taking successive commutator subgroups. Let H be the smallest of the subgroups $[\dots[[G, g], g], \dots, g]$. Note that if $H = 1$, then g is a left-Engel element and therefore $g \in F(G)$ and there is nothing to prove. In any case, $H = [H, g]$.

Let $N = \langle H^G \rangle$ be the normal closure of H . By construction, the image of g in G/N is a left-Engel element and therefore belongs to the Fitting subgroup of G/N . If we prove that the generalized Fitting height of N is at most h for some number h , then N , being a normal subgroup, is contained in $F_h^*(G)$ and then $g \in F_{h+1}^*(G)$. Since H is subnormal in G , by Lemma 2.1(b) the generalized Fitting height of its normal closure N is the same as that of H . Therefore it suffices to obtain the appropriate estimate of the generalized Fitting height of H .

Consider the group $H\langle g \rangle$. Since the nonsoluble length of $E_n(g)$ does not exceed its generalized Fitting height, by Theorem 1.3 the element g belongs to $R_{(k+1)m(m+1)/2}(G)$. Since $H = [H, g]$, it follows that $H \leq R_{(k+1)m(m+1)/2}(G)$, and since H is a subnormal subgroup, $\lambda(H) \leq (k+1)m(m+1)/2$ by Lemma 2.1(a). It remains to obtain bounds for the Fitting height of the $(k+1)m(m+1)/2 + 1$ soluble factors of the upper nonsoluble series of H , which are g -invariant together with H .

Let $X = Y/Z$ be one of such soluble factors, where Y and Z are g -invariant and normal in H and therefore subnormal in G . Since Y is subnormal, we have $h^*(E_n(g) \cap Y) \leq h^*(E_n(g)) = k$ by Lemma 2.1(a). Then the (soluble) image of $E_n(g) \cap Y$ in Y/Z has Fitting height at most k . Since, clearly, $E_{Y,n}(g) \leq E_n(g) \cap Y$, we obtain that $E_{X,n}(g)$, which is the image of $E_{Y,n}(g)$, also has Fitting height at most k .

Consider the ‘outer’ semidirect product $X \rtimes \langle g \rangle$ (recall that both Y and Z are g -invariant, so this semidirect product is well defined). By applying Theorem 1.1 we obtain that $g \in F_{k+1}(X \rtimes \langle g \rangle)$. Therefore $[X, g] \leq F_{k+1}(X \rtimes \langle g \rangle) \cap X \leq F_{k+1}(X)$. In other words, g acts trivially on $X/F_{k+1}(X)$. Since $H = [H, g]$, it follows that H also acts trivially on $X/F_{k+1}(X)$, that is, $X/F_{k+1}(X)$ is a central section of H . In particular, $X = F_{k+2}(X)$, that is, the Fitting height of X is at most $k+2$.

Thus, the generalized Fitting height of H is at most $((k+1)m(m+1)/2 + 1)(k+2) + (k+1)m(m+1)/2 = ((k+1)m(m+1) + 2)(k+3)/2 - 1$, as required. \square

7. FINAL REMARKS AND CONJECTURES

We conjecture that the results of Theorems 1.2 and 1.3 can be strengthened by removing the dependence on the order of the element. In fact, we have quite precise conjectures, with best-possible values. For generalized Fitting height we state the following.

Conjecture 7.1. *Let g be an element of a finite group G , and n a positive integer. If the generalized Fitting height of $E_n(g)$ is equal to k , then $g \in F_{k+1}^*(G)$.*

For nonsoluble length we state the following.

Conjecture 7.2. *Let g be an element of a finite group G , and n a positive integer. If the nonsoluble length of $E_n(g)$ is equal to k , then $g \in R_k(G)$.*

As we show below, it is in this strongest form that Conjectures 7.1 and 7.2 can be derived from an affirmative answer to the following question about automorphisms of direct products of nonabelian finite simple groups.

Question 7.3. *Let $S = S_1 \times \cdots \times S_r$ be a direct product of nonabelian finite simple groups, and φ an automorphism of S transitively permuting the factors. Is it true that $E_{S,n}(\varphi) = S$?*

Here, recall, $E_{S,n}(\varphi) = \langle [x, \underbrace{\varphi, \dots, \varphi}_n] \mid x \in S \rangle$, where commutators are taken in the semidirect product $S\langle\varphi\rangle$. In Lemma 3.9 we obtained an affirmative answer to Question 7.3 in the special case where the order of φ is equal to the number of factors r , that is, when the stabilizer of a point in $\langle\varphi\rangle$ in the induced permutational action on $\{S_1, \dots, S_r\}$ is trivial. However, in general Question 7.3 seems rather difficult and remains open. The first step would be to consider the case of φ of prime order, when of course the open question is about a single nonabelian finite simple group and its automorphism of prime order. A significant headway in this direction was recently made by Robert Guralnick (private communication).

The reduction of Conjectures 7.1 and 7.2 to an affirmative answer to Question 7.3 can be conducted simultaneously based on the following proposition.

Proposition 7.4. *Let α be an automorphism of a finite group G such that $G = [G, \alpha]$. Assume that Question 7.3 has an affirmative answer. Then $E_{G,n}(\alpha) = G$.*

Proof. Let $E = E_{G,n}(\alpha)$ for brevity. Let G be a counterexample of minimal order, and let M be a minimal α -invariant normal subgroup of G . Then $G = ME$ and $M \not\leq E$.

We need to consider two cases: M can be an elementary abelian q -group for a prime q , or a direct product of isomorphic non-abelian simple groups.

If M is an elementary abelian q -group, then the proof proceeds in exactly the same fashion as in the proof of Proposition 2.2 in the soluble case. Then $E \cap M = 1$ by minimality of M because M is abelian and $G = ME$, and so on.

Thus, let $M = S_1 \times \cdots \times S_r$, where the S_i are isomorphic non-abelian simple groups. Suppose that $M \neq C_M(\alpha)$. By the affirmative answer to Question 7.3 every product over an orbit of α in the permutational action on $\{S_1, \dots, S_r\}$ is contained in E whenever α acts on this product nontrivially. Let T be the product of those S_i that are contained in E . Then T is normal in $E\langle\alpha\rangle$, since if $S_i \leq E$ and $x \in E\langle\alpha\rangle$, then $S_i^x \leq E$ and S_i^x is again one of the S_u . Since T is also normal in M , we obtain that T is normal in $ME\langle\alpha\rangle = G\langle\alpha\rangle$. By minimality of M then $M = T \leq E$, a contradiction. Thus, $M = C_M(\alpha)$. Since $C_G(\alpha)$ normalizes E , then $[E, M] \leq M \cap E$ and $M \cap E$ is normal in M and therefore in $ME\langle\alpha\rangle = G\langle\alpha\rangle$. Since $M \not\leq E$, by minimality of M we have $M \cap E = 1$ and therefore, $[E, M] = 1$. As a result, $G = M \times E\langle\alpha\rangle$. This contradicts the hypothesis $G = [G, \alpha]$. \square

Theorem 7.5. *Conjectures 7.1 and 7.2 are true if Question 7.3 has an affirmative answer.*

Proof. As in the proofs of Theorems 1.1 and 1.2, consider the subnormal g -invariant subgroups

$$[\dots[[G, g], g], \dots, g],$$

and let H be the smallest of these subgroups. Note that if $H = 1$, then g is a left-Engel element and therefore $g \in F(G)$ and there is nothing to prove. In any case, $H = [H, g]$.

Let $N = \langle H^G \rangle$ be the normal closure of H . By construction, the image of g in G/N is a left-Engel element and therefore belongs to the Fitting subgroup of G/N . If we prove that the generalized Fitting height of N is at most k , then N , being a normal subgroup,

is contained in $F_k^*(G)$ and then $g \in F_{k+1}^*(G)$, as required. Similarly, if we prove that the nonsoluble length of N is at most k , then as a normal subgroup N is contained in $R_k(G)$ and then $g \in R_k(G)$, as required. Since H is subnormal in G , by Lemma 2.1(b) the generalized Fitting height and the nonsoluble length of its normal closure N are the same as those of H . Therefore it suffices to estimate these parameters of H . But $H = [H, g] \leq E_n(g)$ by Proposition 7.4. Since H is subnormal in G , the generalized Fitting height and the nonsoluble length of H do not exceed the same parameters of $E_n(g)$. The result follows. \square

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